## Formal Logic

Background

This topic deals with Logic (rational inquiry). Two main aims are to:

* Learn a new language, the language of first-order logic (FOL).
* Learn about the notion of logical consequence, and about how one goes about establishing whether some claim is or is not a logical consequence of other accepted claims

Software for this course:

|  |  |
| --- | --- |
| Software | Use |
| Tarski’s World | Atomic languages |
| Fitch | Proofs |
| Boole | ??? |

**Lesson 0**

First-Order Logic

This is a symbolic, artificial language of science used in rational inquiry. It incorporates elements of human languages to precisely explain, without ambiguity, informal notions like grammaticality, meaning, truth and proof.

Example

Atomic Languages

Before we can use FOL, we need to understand how to construct atomic sentences, the simplest sentences in FOL.

Atomic Language

**name**

e.g. *a*

**predicate**

e.g. *Cube()*

In FOL, every symbol comes with a fixed “arity”, a number that tells you how many names it needs to form an atomic sentence.

In Tarski’s world (a tool for interpreting logic), the following atomic sentences are used:

|  |  |  |
| --- | --- | --- |
| Atomic Sentence | Arity | Interpretation |
| Tet(a) | 1 | *a* is a tetrahedron |
| Cube(a) | 1 | *a* is a cube |
| Dodec(a) | 1 | *a* is a dodecahedron |
| Small(a) | 1 | *a* is small |
| Medium(a) | 1 | *a* is medium |
| Large(a) | 1 | *a* is large |
| SameSize(a,b) | 2 | *a* is the same size as *b* |
| SameShape(a,b) | 2 | *a* is the same shape as *b* |
| Larger(a,b) | 2 | *a* is larger than *b* |
| Smaller(a,b) | 2 | *a* is smaller than *b* |
| SameRow(a,b) | 2 | *a* is in the same row as *b* |
| Adjoins(a,b) | 2 | *a* and *b* are located on adjacent (but not diagonally) squares |
| LeftOf(a,b) | 2 | *a* is located nearer to the left edge of the grid than *b* |
| RightOf(a,b) | 2 | *a* is located nearer to the right edge of the grid than *b* |
| FrontOf(a,b) | 2 | *a* is located nearer to the front edge of the grid than *b* |
| BackOf(a,b) | 2 | *a* is located nearer to the back of the grid than *b* |
| Between(a,b,c) | 3 | *a*, *b* and *c* are in the same row, column,  or diagonal, and *a* is between *b*  and *c* |

In FOL, we can also use constructions of predicates/functions to form more complex terms:

Example

father(father(max))

*Max’s Grandfather*

**Lesson 1**

Sound Arguments

Logicians use logical consequence to link together conclusions (statements) to premises (preceding statements).

Example

Cube(a) *a* is a cube

a = b *a* is the same object as *b*

therefore, Cube(b) *b* is a cube

Identity Relations

Four important principles that hold of the identity relation:

=**Elim**: **indiscernibility of identicals. IDENTITY**

If b = c, then whatever holds of b holds of c.

=**Intro**: **Reflexivity of identity REFLEXIVE**

b = b, is always true in FOL

**Symmetry of Identity SYMMETRY**

If b = c, then whatever holds of b holds of c.

**Transitivity of Identity. TRANSIVITY**

If b = c, then whatever holds of b holds of c.

Example: using a Fitch bar to present proofs

1. Cube(c)
2. c = b
3. Cube(b)

= **Elim**: 1, 2

How to give informal proofs using arguments

Example

RightOf(b,c)

LeftOf(d,e)

b = d

LeftOf(c,e)

We are told that b is to the right of c.

So c must be to the left of b, since right of and left of are inverses of one another.

And since b = d, c is left of d, by the indiscernibility of identicals.

But we are also told that d is left of e, and consequently c is to the left of e, by the transitivity of left of.

This is our desired conclusion.

How to give formal proofs using arguments

Example

1. a = b
2. a = a
3. b = a = **Intro**

= **Elim**: 1, 2

**Lesson 2**

Boolean Connectives

|  |  |  |
| --- | --- | --- |
| Symbol | Alternatives | Description |
|  | Not | Negation. |
|  | And | Conjunction. |
|  | Or | Disjunction. |
|  | Then | Implication. |
|  | Equals | Equivalence. |

Below are the game rule’s in Tarski’s world.

Game rule for Negation ()

|  |  |
| --- | --- |
| P | P |
| T | F |
| F | T |

We avoid using double negatives, it usually doesn’t make a difference.

If you commit to the truth of P, you commit to the falsity of P.

Game rule for Conjunction ()

False unless both conjuncts are true

|  |  |  |
| --- | --- | --- |
| P | Q | P Q |
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

If you commit to the truth of P ^ Q then you have implicitly committed yourself to the truth of each of P and Q

Game rule for Disjunction ()

True unless both disjuncts are false

|  |  |  |
| --- | --- | --- |
| P | Q | P Q |
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

If you commit yourself to the truth of P Q, then Tarski's World will make you live up to this by committing yourself to the truth of one or the other

Game rule for Implication ()

True, unless antecedent is true when consequent is false

|  |  |  |
| --- | --- | --- |
| P | Q | P Q |
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

Game rule for Implication ()

True, unless antecedent is true when consequent is false

|  |  |  |
| --- | --- | --- |
| P | Q | P Q |
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

Summary

|  |  |  |  |
| --- | --- | --- | --- |
| Form | Your commitment | Player to move | Goal |
| T | T | You  Tarski’s World | Choose one of P,Q that is true |
| T | F | Tarski’s World  you | Choose one of P,Q that is false |
| F | either | - | Replace P by P and switch commitment |

**Lesson 3**

Formal Proofs

**Conjunction Rules**

Conjunction Elimination ( Elim)

If you have a conjunction in a proof, you may enter on a new line, any of it’s conjuncts.

Example

Cube(a) Large(a)

Cube(a)

Cannot be used if it is embedded as part of a larger sentence (e.g. where the proof has negation in it)

(Cube(a) Large(a))

(Cube(a)

Conjunction Introduction ( Intro)

If you have several sentences in a proof, you may enter on a new line, their conjunction. May occur in any order

Example

1. P Q
2. Q R

3. P Elim: 1

4. R Elim: 2

5. P R Intro: 3, 4

**Disjunction Rules**

Disjunction Introduction ( Intro)

If you have a sentence on a line in a proof, you may enter on a new line, any disjunction of which it is a disjunct.

Example

1. P
2. Q
3. P Q Intro: 1,2

Disjunction Elimination ( Elim)

Corresponds to the method of proof by cases. It incorporates the formal device of a ***subproof***.

Example

1. (A B) (C D)

2. A B

3. B Elim: 2

4. B D Intro: 3

5. C D

6. D Elim: 5

7. B D Intro: 6

1. B D Elim: 1,2 -4, 5, -7

**Negation Rules**

Negation Elimination Introduction ( Elim)

This simple rule allows us to eliminate “double negations”

Example

1. A
2. A Elim: 1

Remember:

A screenshot of a cell phone

Description automatically generated

Remember:

The Up tack or falsum () is a logical constant denoting a false proposition in logic, often called "falsum" or "absurdum". It is always assigned the truth value of F.

Negation Introduction ( Intro)

A version of the method of indirect proof, or proof by contradiction.

If an assumption made leads to , you may close the subproof and derive as a conclusion the negation of the sentence that was the assumption.

Example

1. P P

2. P Elim: 1

3. P Elim: 2

4. Intro: 2,3

5. (P P) Intro: 4

Introduction ( Intro)

Lines in this proof must be in the form P P (i.e. contradictory).

To use this rule, the two sentences must be identical (symbol for symbol), except for the negation sign at the beginning of one of them.

Example

1. Tet(a) Teb(b)
2. Tet(a) Teb(b)
3. Tet(a) Elim: 2
4. Tet(b) Elim: 2

5. Tet(a)

6. Intro 3,5

7. Tet(b)

8. Intro: 4,7

9. Elim: 1,5-6,7-8