## Formal Logic

Background

This topic deals with Logic (rational inquiry). Two main aims are to:

* Learn a new language, the language of first-order logic (FOL).
* Learn about the notion of logical consequence, and about how one goes about establishing whether some claim is or is not a logical consequence of other accepted claims

Language, Proof and Logic is an educational software package used to teach formal logic using a textbook and four software programs.

Software for this course:

|  |  |  |  |
| --- | --- | --- | --- |
| Software | Named after | Use | Description |
| Tarski’s World | Alfred Tarski | Atomic languages | Teaches the basic first-order language and its semantics using a model theoretic-like approach, where the "world" consists of a little grid and some simple objects |
| Fitch | Frederic Brenton Fitch | Proofs | Fitch-style calculus for checking first-order proofs |
| Boole | George Boole | Boolean connectives | Construction of truth tables and related notions (tautology, tautological consequence, etc.) |

**Lesson 0**

First-Order Logic

This is a symbolic, artificial language of science used in rational inquiry. It incorporates elements of human languages to precisely explain, without ambiguity, informal notions like grammaticality, meaning, truth and proof.

Example

Atomic Languages

Before we can use FOL, we need to understand how to construct atomic sentences, the simplest sentences in FOL.

Atomic Language

**name**

e.g. *a*

**predicate**

e.g. *Cube()*

In FOL, every symbol comes with a fixed “arity”, a number that tells you how many names it needs to form an atomic sentence.

In Tarski’s world (a tool for interpreting logic), the following atomic sentences are used:

|  |  |  |
| --- | --- | --- |
| Atomic Sentence | Arity | Interpretation |
| Tet(a) | 1 | *a* is a tetrahedron |
| Cube(a) | 1 | *a* is a cube |
| Dodec(a) | 1 | *a* is a dodecahedron |
| Small(a) | 1 | *a* is small |
| Medium(a) | 1 | *a* is medium |
| Large(a) | 1 | *a* is large |
| SameSize(a,b) | 2 | *a* is the same size as *b* |
| SameShape(a,b) | 2 | *a* is the same shape as *b* |
| Larger(a,b) | 2 | *a* is larger than *b* |
| Smaller(a,b) | 2 | *a* is smaller than *b* |
| SameRow(a,b) | 2 | *a* is in the same row as *b* |
| Adjoins(a,b) | 2 | *a* and *b* are located on adjacent (but not diagonally) squares |
| LeftOf(a,b) | 2 | *a* is located nearer to the left edge of the grid than *b* |
| RightOf(a,b) | 2 | *a* is located nearer to the right edge of the grid than *b* |
| FrontOf(a,b) | 2 | *a* is located nearer to the front edge of the grid than *b* |
| BackOf(a,b) | 2 | *a* is located nearer to the back of the grid than *b* |
| Between(a,b,c) | 3 | *a*, *b* and *c* are in the same row, column,  or diagonal, and *a* is between *b* and *c* |

**Lesson 1**

Formal properties

**Properties of Binary Operations**

Identity:

* Properties of one expression can translate to another if the expressions are equivalent

Inverse:

* The opposite.

Idempotent:

* This means that no matter applied multiple times without changing the result beyond the initial application.

**Rules of replacement (**"can be replaced in a logical proof with".)

Commutativity (Symmetrical) Property:

* P Q Q P
* P Q Q P

Associativity:

* P (Q R) (P Q) R
* P (Q R) (P Q) R

Distributivity:

* P (Q R) (P Q) (P R)
* P (Q R) (P Q) (P R)

Double Negation:

* P P
* P P

De Morgan’s Law: Negate all the names AND connectives

* (P Q) P Q
* (P Q) P Q

Transposition:

* (P Q) Q P

Exportation:

* (P Q) R P (Q R)

Material implication:

* (P Q) Q

Tautology:

* P P P (idempotency of conjunction)
* P P P (idempotency of disjunction)

Conjunction is **idempotent**.

Conjunction is **commutative** (symmetrical).

**Lesson 2**

Sound Arguments

Logicians use logical consequence to link together conclusions (statements) to premises (preceding statements).

Example

Cube(a) *a* is a cube

a = b *a* is the same object as *b*

therefore, Cube(b) *b* is a cube

Identity Relations **I–R–S-T**

(Also known as logic axioms for identity)

Four important principles that hold of the identity relation:

**IDENTITY**

=**Elim**: **indiscernibility of identicals.**

If b = c, then whatever holds of b holds of c.

=**Intro**: **Reflexivity of identity**

b = b, is always true in FOL

**Symmetry of Identity**

If b = c, then c = b

**Transitivity of Identity.**

If a = b, and b = c, then a = c

substitution

**REFLEXIVITY**

x = x

**SYMMETRY**

x = y y = x

**TRANSIVITY**

x = y y = z

Example: using a Fitch bar to present proofs

1. Cube(c)
2. c = b
3. Cube(b)

= **Elim**: 1, 2

How to give informal proofs using arguments

Example

RightOf(b,c)

LeftOf(d,e)

b = d

LeftOf(c,e)

We are told that b is to the right of c.

So c must be to the left of b, since right of and left of are inverses of one another. And since b = d, c is left of d, by the indiscernibility of identicals. But we are also told that d is left of e, and consequently c is to the left of e, by the transitivity of left of.

This is our desired conclusion.

**Lesson 3**

Logical Possibility (propositional Equivalence)

**A close up of a sign

Description automatically generatedTautology.**

The saying of the same thing twice over in different words

A tautology is a formula which is "**always true"**

The negation of a tautology is a TT-Contradiction

P P

A close up of a sign

Description automatically generated

**Contradiction. TT-Contradiction**

A combination of statements, ideas, or features which are opposed to one another.

A tautology is a formula which is "**always false"**

The negation of a TT-Contradiction is a Tautology

P P



**Contingency. TT-Contingency**

Neither a tautology nor contradiction.

A tautology is a formula which is "**always false"**

P Q R

In Tarski’s world, we use the truth table to show that certain sentences cannot possibly be false. In this world, the truth table method works only in one direction: when it says that a sentence is logically necessary, then it is.

A screenshot of a cell phone

Description automatically generatedTypes of possibilities

iff at least one row on its truth table reads T

<https://www.ocf.berkeley.edu/~brianwc/courses/logic/notes04.html>

|  |  |  |
| --- | --- | --- |
| **1** | **TT-Possible** | iff at least one row on its truth table reads T under it’s main connective |
|  | Tautology | iff every row on its truth table reads T under it’s main connective |
| **2** | **TW-Possible** | iff it is true in at least one world in Tarski's World (can be built in Tarski’s World) |
| **3** | **TW-necessary** | iff it is true in every world in Tarski's World |
|  | tautologically equivalent | iff every row of their joint truth table assigns the same values to each |

**Lesson 4**

Boolean Connectives

|  |  |  |  |
| --- | --- | --- | --- |
| Symbol | Meaning | Alternatives | Description |
|  | Not |  | Negation. |
|  | And |  | Conjunction. |
|  | Or |  | Disjunction. |

Below are the game rule’s in Tarski’s world.

Game rule for Negation () (01)

|  |  |
| --- | --- |
| P | P |
| T | F |
| F | T |

–

We avoid using double negatives, it usually does not make a difference.

If you commit to the truth of P, you commit to the falsity of P.

Game rule for Conjunction () (1000)

False unless both conjuncts are true

|  |  |  |
| --- | --- | --- |
| P | Q | P Q |
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

If you commit to the truth of P ^ Q then you have implicitly committed yourself to the truth of each of P and Q

Game rule for Disjunction () (1110)

True unless both disjuncts are false

|  |  |  |
| --- | --- | --- |
| P | Q | P Q |
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

If you commit yourself to the truth of P Q, then Tarski's World will make you live up to this by committing yourself to the truth of one or the other

When two sentences are logically equivalent, each is a logical consequence of

the other.

**Lesson 5**

Conditionals (Logical Consequence)

|  |  |  |  |
| --- | --- | --- | --- |
| Symbol | Meaning | Alternatives | Description |
|  | If… then |  | Material Conditional (Implication) |
|  | Iff  OR  “Just in case” |  | Material biconditional (Equivalence). "can be replaced in a logical proof with". |

Logical truth - A sentence is a logical consequence of a set of sentences if it is impossible for that sentence to be false when all the sentences in the set are true.

Game rule for Implication () (1011)

True, unless antecedent is true when consequent is false

|  |  |  |
| --- | --- | --- |
| P | Q | P Q |
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

In Tarski’s World, P Q is another way of saying ¬P Q.

example

If Max is home then Claire is at the library

Home(max) Library (claire)

Game rule for Biconditional () (1001)

True if and only if P and Q have the same truth value

|  |  |  |
| --- | --- | --- |
| P | Q | P Q |
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

example

Max is home if and only if Claire is at the library

Home(max) Library (claire)

Remember:

Valid/Sound Argument

*Logical consequence of its premises*

true premise

true conclusion

true premise

true premise

Invalid Argument

*Not a logical consequence of its premises*

true premise

false conclusion

true premise

true premise

Valid/Unsound Argument

*Not assured that conclusion is true*

false premise

??? conclusion

true premise

true premise

An argument can be valid but unsound if the conclusion is true, but the premises are not all true.

Summary

|  |  |  |  |
| --- | --- | --- | --- |
| Form | Your commitment | Player to move | Goal |
| T | T | You  Tarski’s World | Choose one of P,Q that is true |
| T | F | Tarski’s World  you | Choose one of P,Q that is false |
| F | either | - | Replace P by P and switch commitment |

**Lesson 6**

Formal Proofs: Methods of Proof

Boolean connectives also give rise to two entirely new methods of proof:

* Proof by cases (disjunction elimination)
* Indirect proof: Proof by contradiction (tautologies)

**Proof by cases**

In a proof by cases, we must cover all possible cases that arise in a theorem.

Example: prove if n is an integer

We know that Case 1 Case 2 Case 3

Case 1: , so it follows that

Case 2:

, so it follows that

Case 3:

, so it follows that

Since our inequality is true for all possible cases, we can conclude for all integers

Structure:

1. We know (n cases)
2. Our goal is to prove S
3. If is the case, S follows

OR

1. If is the case, S follows

OR

1. If is the case, S follows

Example

(Home(max) Happy(carl)) (Home(Claire) Happy(scruffy))

We want to prove that either Carl or Scruffy is happy

Happy(carl) Happy(scruffy)

Case 1: Home(max) Happy(carl)

So it follows that Happy(carl) Happy(scruffy)

Case 2: Home(claire) Happy(scruffy)

So it follows that Happy(carl) Happy(scruffy)

**Proof by contradiction** *reductio ad absurdum.*

Remember: Contradiction is a combination of statements, ideas, or features which are opposed to one another.

In proof by contradiction, we assume a proposition is not true. Then through premise and logic, find a contradiction that shows our original premise must have been incorrect

Structure: **proposition**

1. Our goal is to prove S
2. Assume S
3. Prove a contradiction

Structure: **implication (material conditional)**

1. Our goal is to prove P Q
2. Assume P and Q are true
3. Prove a contradiction that shows P Q

OR

1. Prove a contradiction that shows P Q

Example: prove that is irrational

is irrational = S

Therefore, assume that is rational

Therefore, there exist 2 integers where and have no common factors.

Although, there exist no integers where the above is true. Hence, is irrational via contradiction

**Lesson 7**

Formal Proofs: Rules of inference

**Conjunction Rules**

Conjunction Elimination ( Elim)

If you have a conjunction in a proof, you may enter on a new line, any of it’s conjuncts.

Example

Cube(a) Large(a)

Cube(a)

Cannot be used if it is embedded as part of a larger sentence (e.g. where the proof has negation in it)

(Cube(a) Large(a))

(Cube(a)

Conjunction Introduction ( Intro)

If you have several sentences in a proof, you may enter on a new line, their conjunction. May occur in any order

Example

1. P Q
2. Q R

3. P Elim: 1

4. R Elim: 2

5. P R Intro: 3, 4

**Disjunction Rules**

Disjunction Introduction ( Intro)

If you have a sentence on a line in a proof, you may enter on a new line, any disjunction of which it is a disjunct.

Example

1. P
2. Q
3. P Q Intro: 1,2

Disjunction Elimination ( Elim)

Corresponds to the method of proof by cases (cover every single possibility in the theorem). It incorporates the formal device of a ***subproof***.

Example

1. (A B) (C D)

2. A B

3. B Elim: 2

4. B D Intro: 3

5. C D

6. D Elim: 5

7. B D Intro: 6

1. B D Elim: 1,2 -4, 5, -7

**Negation Rules**

Negation Elimination Introduction ( Elim)

This simple rule allows us to eliminate “double negations”

Example

1. A
2. A Elim: 1

To reduce “double negations” we use negation normal form (NNF)

Example

¬¬(A ∨ ¬B) ∧ ¬(¬¬A ∧ ¬B)

= (A ∨ ¬B) ∧ ¬(¬¬A ∧ ¬B)

= (A ∨ ¬B) ∧ ¬(A ∧ ¬B)

= (A ∨ ¬B) ∧ (¬A V B)

A screenshot of a cell phone

Description automatically generatedRemember:

Remember:

The Up tack or falsum () is a logical constant denoting a false proposition in logic, often called "falsum" or "absurdum". It is always assigned the truth value of F.

Negation Introduction ( Intro)

A version of the method of indirect proof, or proof by contradiction.

If an assumption made leads to , you may close the subproof and derive as a conclusion the negation of the sentence that was the assumption.

Example

1. P P

2. P Elim: 1

3. P Elim: 2

4. Intro: 2,3

5. (P P) Intro: 4

Negation Introduction ( Intro)

Lines in this proof must be in the form P P (i.e. contradictory).

To use this rule, the two sentences must be identical (symbol for symbol), except for the negation sign at the beginning of one of them.

Example

1. Tet(a) Teb(b)
2. Tet(a) Teb(b)
3. Tet(a) Elim: 2
4. Tet(b) Elim: 2

5. Tet(a)

6. Intro 3,5

7. Tet(b)

8. Intro: 4,7

9. Elim: 1,5-6,7-8

TODO:  
Introduction: Intro

Elimination: Elim

Conditional Intro

Conditional Elimination

**Lesson 8**

Informal methods of proof: Rules of Inference

**Conditional elimination:** “method of affirming”

P implies Q and P is asserted to be true, therefore Q must be true.

P and P → Q, you may infer Q.

Example: “John will prove a theorem only if he isn't very tired. He slept very well last night, so he'll prove a theorem.”

P: John is not tired

Q: John will prove a theorem

**Biconditional elimination:** “method of affirming for the biconditional”

From P and P ↔ Q , you may infer Q.

From P and Q ↔ P , you may infer Q.

**Contraposition**

P → Q ⇔ ¬Q → ¬P

**“conditional – disjunction” equivalence**

P → Q ⇔ ¬P ∨ Q

**“negated conditional” equivalence**

¬(P → Q) ⇔ P ∧ ¬Q

**“biconditional – conjunction” equivalence**

¬(P → Q) ⇔ P ∧ ¬Q

**“biconditional – disjunction” equivalence**

P ↔ Q ⇔ (P → Q) ∧ (Q → P)

**Lesson 9**

Quantification

We have at our disposal at least two languages:

1. A formal language. An abstract, truth-functional language (Propositional Logic: PL)
2. A natural language (English)

**Translations: Atomic Well-Formed-Formulae (WFF’s)**

A WFF is a string of symbols that is part of the formal language.

Natural language prioritises communication and flexibility

Formal languages prioritise precision and rigidity.

To create a WFF, we need:

The correct components

The correct order

Example: Doctor(x) ∧ Smart(x)

**Quantifier Symbols:**

Universal quantifier ()

Express universal claims, those we express in English using quantified phrases like *everything, each thing all things,* and *anything*.

Example:

Every doctor is smart.

x(Doctor(x) → smart(x))

Existential Quantifier ()

Express existential claims, those we express in English

thing, a, an using such phrases as *something, at least one thing, a,* and *an*.

Example:

Some doctor is smart.

x(Doctor(x) → smart(x))